THE WAVE STATISTICAL THEORY OF ALPHA-DISINTEGRATION

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ABSTRACT The present paper is an attempt to study the phenomenon of alpha-decay from a completely new standpoint. All the previous theories are based on the leakage hypothesis, first introduced by Gamow (1928). The recent experiments of Chang (1946) are, however, "in serious disagreement" with the above theories. So the need of reformulation of the above hypothesis has been stressed by Chang and others.

According to the present theory the alpha particles do not leak through the potential barrier by some unknown mechanism, but actually surmount the barrier as required by classical mechanics. And on coming out of the nucleus they are acted upon by a retarding attraction field of Yukawa type, which will obviously reduce the initial high energy to the low observed value. Instead of the usual wave equation, we have taken those from the principles of wave statistics, given by Kar (1940), which give a physical interpretation for the internal mechanism of disintegration.

The chief advantages of the present theory are:

(1) An *explicit* formula for the nuclear radius, *independent of* λ, has been obtained for the first time [*vide* Eq. (36)]. The values of nuclear radii have been calculated from the observed energy values (Table I). The values thus obtained are unique for given Z-values, unlike the numerous values given by the previous theories.

(2) The maximum values of the extra-nuclear Yukawa type field have been calculated from the formula obtained [*vide* Eq. (35.1)]. The order of the magnitudes obtained agree with the values of nuclear binding energies calculated from mass-defects, etc.

(3) The formula for disintegration constant λ differs in the exponential part from the Gamow-factor, in having the internal energy of the alpha particle instead of its observed value. This will explain the discrepancy between the previous theories and the recent experimental results.

(4) It explains qualitatively the general decrease in the number of spectral lines as we go up a radioactive series.

INTRODUCTION

The first attempt to give a wave-mechanical treatment of the problem of alpha-decay was made by Gamow (1928), and Condon and Gurney (1928). The difficulty in explaining the difference between the observed low energy of the emitted alpha particles and the high energy of the potential barrier at the surface of the nucleus (indicated by scattering experiments) led them to introduce the well-known hypothesis of leakage. It is also assumed that the energy of the alpha particles inside the nucleus is complex and that the imaginary part is the cause of the leakage (Gamow, 1937).

Since then attempts have been made by Sexl (1929), Kar (1933), Bethe (1937), Preston (1946) and many others* to study the phenomenon of alpha-decay. Each attempt was initiated with the purpose of removing the lack of mathematical rigour at places in Gamow's theory noticed by the different authors. But the fundamental hypothesis of leakage and the complex energy for a real particle were retained throughout by others except by Kar (1933), who gave up the latter hypothesis though retained the former All these theories have given almost the same formula for disintegration constant λ, as obtained by Gamow.

The experimental agreement of Gamow's formula was found to be more or less satisfactory. However, recently with the collection of more accurate data, Berthellot (1942) has suggested that for better agreement, comparison with the experimental Geiger-Nuttal curve should be made for isotopes only. More recently, objections have also been raised by Chang (1946), who has discovered low energy spectral lines for alpha-rays from Ra, RaF, etc. According to him, the existing theory of alpha-disintegration is "in serious disagreement" with his experimental results. For, "the theoretical intensity varies much more rapidly than the observed intensity." Another serious objection, pointed out by Chang and also admitted by Gamow (1949), is that unacceptably large spin changes, as calculated from Gamow's theory, occur in the case of normal-normal and normal-high alpha transformations. He has also pointed out that if the internal energy of the alpha particles be supposed to be higher than the observed value, in contradiction to the fundamental hypothesis of leakage, the discrepancy can be reconciled. Hence while stressing the need of "reformulation" of the current decay theory, Chang (1946) has suggested a modification in a qualitative way. He assumes that penetration occurs through the barrier at a higher energy level than the corresponding observed low energy and that the original difficulty of energy difference can be explained by postulating some sort of semi-static interaction between the outgoing alpha particle and the product nucleus, resulting in the transfer of the balance energy back to the residual nucleus. However, as the exact mechanism of this transfer of energy is not known the position remains more or less as vague as in Gamow's conception of leakage.

SECTION 1

In the proposed theory, we shall suppose that the alpha particle does not leak through the barrier due to some unknown mechanism but actually surmounts the potential hill with energy higher than or just sufficient to overcome it, as required by classical mechanics. We also assume that as the alpha particle just comes out of the nucleus, a retarding attraction field of Yukawa type is at once set up between the outgoing alpha particle and the product nucleus. This extra-nuclear attraction field would obviously reduce

* See references.

the initial high energy of the alpha particle to the observed low value. As, however, this short range attraction force is operative only when the alpha particle has left the nucleus, the minimum energy with which it can come out from inside the nucleus is equal to the peak energy of the Coulomb field, *i.e.*, $2z^*e^2/r_0$, where r_0 = nuclear radius, and z^* = charge number of the product nucleus. If the attraction potential $U(r)$ is effective upto a short distance, reducing to an insignificant value at r_1 (say), no further reduction in the energy of the alpha particle can take place beyond this range. Thus the total energy of the alpha particle at r_1 is equal to the final observed kinetic energy.

Now from what has been stated above, the whole phase space may be divided into three regions, namely.

Region 1. $0 < r < r_0$, *i.e.*, the interior of the nucleus.

In this region the potential energy U' may be taken to be constant. Thus $E' + U' = E_1$, the total kinetic energy inside the nucleus.

Region 2. $r_0 < r < r_1$, *i.e.*, the extent of the short range force.

In this region the Coulomb potential $V = 2z^*e^2/r$. The short range attraction potential actually varies from $U(r_0)$ at r_0 to $U(r_1) = 0$ at r_1. We shall, however, assume for simplicity the mean constant potential U_0 in the wave equation for this region.

Region 3. $r_1 < r < \infty$, *i.e.*, outside the influence of the short range force.

In this region, the Coulomb potential $V = 2z^*e^2/r$, $U(r) = 0$, kinetic energy $= (E_3 - V)$, where E_3 is the total energy of the alpha particle, and is obviously the observed alpha energy, $\frac{1}{2}Mv^2$.

FIG. 1(*a*)

→ indicates α-particle, V = coulomb potential. I, II, III are the respective regions. ΔE = difference of internal and observed energy of the α-particle $U(r)$ = Extra-nuclear attraction potential of nuclear force type. U_0, its mean (rectangular, outside) U' = Internal rectangular potential.

FIG. 1(*b*)

E = Internal α-energy = observed α-energy U' = Attraction potential (internal, rectangular)

The distribution of fields assumed in the proposed theory is shown in Fig. 1(*a*). For comparison the distribution assumed by the previous workers is given in Fig. 1(*b*),

Region 1. In this region the wave equation from the principles of wave statistics, is given by

$$\Delta\chi_1 + \frac{8\pi^2 ME_1}{h^2}\left(1 + \frac{b^2h^2}{16\pi^2E_1{}^2}\right)\chi_1 = 0$$

where χ_1 represents the wave function for the alpha particle for q-space. This wave equation differs from that of Schrodinger in having the term $b^2h^2/16\pi^2E_1{}^2$, where b is the damping co-efficient. It is to be noted that for mechanism of disintegration, Kar (1940) has suggested from hydrodynamical analogy that when viscosity develops in the highly dense but compressible nuclear phase fluid, alpha particle comes out as a result of damping. Thus damping factor b provides a physical interpretation for disintegration, and b is defined by $b.D_s.dt = dD_s$, where D_s is the number density of alpha particles just inside the surface of the nucleus. Now since the emission is assumed to be spherically symmetrical, we have $l = 0$, and so the radial component of the wave function χ_1 for region 1 satisfies

$$\frac{d^2R_1}{dr^2} + \frac{2}{r}\cdot\frac{dR_1}{dr} + \frac{8\pi^2M.E_1}{h^2}\left(1 + \frac{b^2h^2}{16\pi^2E_1{}^2}\right).R_1 = 0 \qquad \ldots \quad (1)$$

Putting $R_1 = \dfrac{R'_1}{r}$, in Eq. (1) we have

$$\frac{d^2R'_1}{dr^2} + \frac{8\pi^2ME_1}{h^2}\left(1 + \frac{b^2h^2}{16\pi^2E_1{}^2}\right)R_1' = 0 \qquad \ldots \quad (1.1)$$

The solution of Eq. (1.1) gives

$$R_1 = \frac{R'_1}{r} = \text{const.}\ \frac{\sin \alpha_1 r}{r} \qquad \ldots \quad (1.2)$$

where

$$\alpha_1{}^2 = \frac{8\pi^2ME_1}{h^2}\left(1 + \frac{b^2h^2}{16\pi^2E_1{}^2}\right) \qquad \ldots \quad (1.3)$$

Regions 2 *and* 3. Outside the nucleus b is absent. Consequently, on taking corresponding energies, we have the wave equation for the region 2 $(r_0 < r < r_1)$ as

$$\Delta\chi_2 + \frac{8\pi^2M}{h^2}\,(E_2 + U_0 - V(r)\,).\chi_2 = 0 \qquad \ldots \quad (1.4)$$

and for the region 3 $(r_1 < r < \infty)$ as

$$\Delta\chi_3 + \frac{8\pi^2M}{h^2}\,(E_3 - V(r)\,).\chi_3 = 0 \qquad \ldots \quad (1.5)$$

These are identical with the usual wave equations except for the term U_0 in region 2. Therefore the modified R-equations for these external regions are

$$\frac{d^2R_2}{dr^2}+\frac{2}{r}\cdot\frac{dR_2}{dr}+\frac{8\pi^2M}{h^2}(E_2+U_0-V(r)).R_2=0 \quad \ldots (2)$$

and

$$\frac{d^2R_3}{dr^2}+\frac{2}{r}\cdot\frac{dR_3}{dr}+\frac{8\pi^2M}{h^2}(E_3-V(r)).R_3=0 \quad \ldots (3)$$

On putting $R_2=\frac{R_2'}{r}$, and $R_3=\frac{R_3'}{r}$, respectively in (2) and (3), we have

$$\frac{d^2R_2'}{dr^2}+\frac{8\pi^2M}{h^2}(E_2+U_0-V(r))\,R_2'=0 \quad \ldots (2.1)$$

and

$$\frac{d^2R_3'}{dr^2}+\frac{8\pi^2M}{h^2}(E_3-V(r)).R_3'=0 \quad \ldots (3.1)$$

where $M=$ mass of alpha particle, $E=$total energy of the α-particle, $V(r)=2z^*e^2/r$, $z^*=z-2$. The subscripts 2, and 3 respectively refer to regions 2 and 3.

To solve Eq. (2.1) we put

$$\left.\begin{array}{l} \rho=\alpha_2 r \\ \alpha_2=4\pi\sqrt{2M(E_2+U_0)}/h \end{array}\right\} \quad \ldots (2.2)$$

where the second line defines α_2.

Therefore Eq. (2.1) becomes

$$\frac{d^2R_2'}{d\rho^2}+\left(\frac{1}{4}-\frac{K_2}{\rho}\right)R_2'=0 \quad \ldots (2.3)$$

where

$$K_2=\frac{4\pi z^*e^2\sqrt{M}}{h\sqrt{2(E_2+U_0)}} \quad \ldots (2.4)$$

Again by substituting

$$\left.\begin{array}{l} \rho=2x \\ K_2=K'_2/2 \end{array}\right\} \quad \ldots (2.5)$$

Eq (2.3) can further be simplified and we have

$$\frac{d^2R_2'}{dx^2}+\left(1-\frac{K_2'}{x}\right)R_2'=0 \quad \ldots (2.6)$$

Similarly from Eq. (3.1) we have evidently

$$\frac{d^2R_3'}{dx'^2}+\left(1-\frac{K_3'}{x'}\right)R_3'=0 \quad \ldots (3.2)$$

where we have put

$$\left.\begin{array}{l} \rho'=\alpha_3 r=2x' \\ \alpha_3=4\pi\sqrt{2ME_3}/h \\ K_3=\dfrac{4\pi z^*e^2\sqrt{M}}{h\sqrt{2E_3}}=\dfrac{K_3'}{2} \end{array}\right\} \quad \ldots (3.3)$$

Now Eqs. (2.6) and (3.2) are standard differential equations. P. Debye (1909) has given a solution by the method of steepest descent, which has been used by Preston (1946). We shall, however, follow the method given by Schlechinger (1900), and quoted by Sexl (1929). Both the methods, of course, lead to the same solution.

It would be useful to give a brief sketch of the method to be followed. Let us assume a solution of (2.6) in the form

$$R'_2 = e^{\int g.dx} \qquad \ldots (4)$$

where

$$g = g_0 + \lambda g_1 + \cdots \qquad \ldots (4.1)$$

in which λ and g_1 are small. On substituting (4) in (2.6) and neglecting the small quantities $\lambda dg_1/dx$ and $\lambda^2 g_1{}^2$, we have

$$g_0' + g_0{}^2 + 2\lambda g_0 g_1 + 1 - \frac{K_2'}{x} = 0 \qquad \ldots (5)$$

where $\qquad g_0' = \dfrac{dg_0}{dx}.$

The following three cases may now be considered :

Case 1. Let x be very small and K'_2 very great. Along with x, r becomes very small. So the solution refers to a region inside the nucleus. Thus the case is apparently invalid for an equation holding for a region where there is Coulomb field.

Case 2. Let x and K_2' be both great Hence from Eq. (5), equating terms of the same order we get

and

$$\left.\begin{aligned} g_0 &= \pm\sqrt{\frac{K_2'}{x} - 1} \\ \lambda g_1 &= -\tfrac{1}{2}.\ \frac{d}{dx}\ .\ln g_0 \end{aligned}\right\} \qquad \ldots (4.3)$$

Using Eq. (4.3) in (4) we have for the solution

$$R_2' = \frac{\text{const.}}{\sqrt[4]{\dfrac{K_2'}{x} - 1}}\ .\ e^{\pm\int\sqrt{\frac{K_2'}{x} - 1}\ .dx} \qquad \ldots (6)$$

Here again the solution (6) may be of two different forms according as,

(a) $\dfrac{K_2'}{x} > 1$, and so putting $\dfrac{K_2'}{x} = \dfrac{1}{\cos^2 u}$, the solution becomes

$$R_2' = \text{const.}\sqrt{\cot u}.\ e^{\pm K_2(2u - \sin 2u)} \qquad \ldots (6.1)$$

or, as,

(b) $\dfrac{K_2'}{x} < 1$, and so putting $\dfrac{K_2'}{x} = \dfrac{1}{\cosh^2 u}$, the solution becomes

$$R_2' = \text{const.} \sqrt{\coth u}.e^{\pm i\{K_2(\sinh 2u - 2u) - \pi/4\}} \qquad \dots (6.2)$$

Case 3. Let x be very great and K_2' finite, so that $\frac{K_2'}{x} \ll 1$. Hence from Eq. (5), on equating terms of the same order we get

$$\left.\begin{aligned} g_0 &= \pm i \\ \text{and} \qquad \lambda g_1 &= -\tfrac{1}{2}.\ \frac{d}{dx}\ .\ \ln.\ g_0 \pm \frac{K'_2}{2ix} \end{aligned}\right\} \qquad \dots (4.4)$$

Therefore, using (4.4) in (4), the solution becomes

$$R'_2 = \text{const.}\ e^{\pm i\left(\frac{a_2}{2}\ r \mp K_2\ \ln.\frac{a_2}{2}.r - \frac{\pi}{4}\right)} \qquad \dots (7)$$

It may easily be shown from (2.2), (2.4) and (2.5) that

$$\frac{K'_2}{x} = \frac{2z^* e^2/r}{E_2 + U_0} \qquad \dots (7.1)$$

Region 2. In this region near r_0, $\frac{2z^*e^2}{r} > E_2 + U_0$, because even for zero initial kinetic energy of the α-particle, $(E_{kin})_{r_0} = E_2 + U(r_0) - \frac{2z^*e^2}{r_0} = 0$ *i.e.* $\frac{2z^*e^2}{r_0} = E_2 + U(r_0)$ and $U(r_0) > U_0$ {Fig. 1 (*a*) and 1 (*b*) }. Then from r_0 outwards as the Coulomb potential $\frac{2z^*e^2}{r}$ decreases, it must be equal to $E_2 + U_0$ at certain value $r = r'$, *i.e.*, at r',

$$\frac{2z^*e^2}{r'} = E_2 + U_0 \qquad \dots (7.2)$$

So for the region 2(*a*) *i.e.*, for $r_0 < r < r'$, from Eq. (7.1), $\frac{K_2'}{x} > 1$, hence for this range we have to take the solution as (vide (6.1)

$$R_2 = \frac{R_2'}{r} = \frac{\text{const}}{r}.\sqrt{\cot u}.\ e^{\pm K_2(2u - \sin 2u)} \qquad \dots (8)$$

Again for the region 2(*b*) *i.e.*, for $r' < r < r_1$, $\frac{2z^*e^2}{r} < E_2 + U_0$ and so from Eq. (7.1) $\frac{K_2'}{x} < 1$. Therefore in this range we have to take the solution as (*vide* (6.2))

$$R_2 = \frac{R'_1}{r} = \frac{\text{const}}{r} . \sqrt{\coth u}\; e^{\pm i\left\{K_2(\sinh 2u - 2u) - \frac{\pi}{4}\right\}} \quad \dots \quad (8.1)$$

If, however, r_1 is sufficiently great, so that $\frac{K'_2}{x} \ll 1$ near r_1, we should take solution as {vide (7) },

$$R_2 = \frac{\text{const.}}{r} . e^{\pm i\left(\frac{\alpha_2}{2} . r - K_2 \ln \frac{\alpha_2}{2} r - \frac{\pi}{4}\right)} \quad \dots \quad (8.2)$$

instead of Eq. (8.1). But as we do not know where r_1 actually falls, we should start by taking both the solutions (8.1) and (8.2) for the region 2 (b), finally rejecting the one found to be inappropriate from other considerations.

Region 3. In this region beyond r_1, $\frac{2z^* e^2}{r}$ is definitely very small, and so we have, $\frac{K_3'}{x'} = \frac{2z^* e^2/r}{E_3} \ll 1$, similar to Eq. (7.1). Therefore we have to take the solution of Eq, (3.2), corresponding to (7) as

$$R_3 = \frac{\text{const}}{r} . e^{\pm i\left(\frac{\alpha_3}{2} . r - K_3 \ln \frac{\alpha_3}{2} r - \frac{\pi}{4}\right)} \quad \dots \quad (9)$$

Boundary Conditions :

Of the boundary conditions of continuity at $r = r_0$, and at $r = r_1$, the latter is useful in giving valuable informations regarding the size of the nucleus, and the magnitudes of the attraction potential $U(r)$ at r_0 in the region 2. So we consider it first. The other boundary condition will be considered in Sec. 2.

The R-components of the wave equations (1.4) and (1.5) satisfy the combined boundary conditions at $r = r_1$, namely,

$$\left\{\frac{d}{dr} . \log R_2\right\}_{r=r_1} = \left\{\frac{d}{dr} . \log R_3\right\}_{r=r_1} \quad \dots \quad (10)$$

As already discussed under Eq. (7.1), R_2 may have two forms near r_1, according as $\frac{K'_2}{x} < 1$ or $\ll 1$. Taking the real part of the first form, given in (8.1) (the unknown constant, in general, being considered complex) we have

$$R_2 = \frac{\text{const.}}{r} . \sqrt{\coth u} . \cos\left\{K_2\left(\sinh 2u - 2u\right) - \frac{\pi}{4} + p_2\right\} \quad \dots \quad (10.1)$$

where p_2 is the unknown epoch. On taking the above value of R_2, the left hand side of (10) becomes

$$-\frac{1}{r_1} + \frac{1}{4 r_1 \left(1 - \frac{\alpha_2 r_1}{4K_2}\right)} - \frac{\alpha_2}{2} \sqrt{1 - \frac{4K_2}{\alpha_2 r_1}} \times \tan\left\{K_2\left(\sinh 2u_1 - 2u_1\right) - \frac{\pi}{4} + p_2\right\} \quad \dots \quad (11)$$

Again taking the real part of the second form of R_2, given in Eq. (8.2) we have

$$R_2=\frac{\text{const.}}{r}\ .\ \cos\left(\frac{\alpha_2}{2}r-K_2\ \ln.\frac{\alpha_2}{2}.r-\frac{\pi}{4}+p_2'\right)$$

where p_2' is the corresponding epoch With this value of R_2, the l. h. s. of (10) becomes

$$-\frac{1}{r_1}-\left(\frac{\alpha_2}{2}-\frac{K_2}{r_1}\right)\times\ \tan\left(\frac{\alpha_2}{2}r_1-K_2\ \ln\frac{\alpha_2}{2}r_1-\frac{\pi}{4}+p_2'\right)\quad ..\quad (11.1)$$

For the right hand side of (10) we take the real part of R_3, given in Eq (9) and get

$$R_3=\frac{\text{const.}}{r}\ .\ \cos\left(\frac{\alpha_3}{2}r-K_3\ \ln.\frac{\alpha_3}{2}r-\frac{\pi}{4}+p_3\right)$$

where p_3 is the unknown epoch. On substituting the above value of R_3, we have for the right hand side of (10)

$$-\frac{1}{r_1}-\left(\frac{\alpha_3}{2}-\frac{K_3}{r_1}\right)\times\tan\left(\frac{\alpha_3}{2}r_1-K_3\ \ln\frac{\alpha_3}{2}r_1-\frac{\pi}{4}+p_3\right)\qquad ..\quad (12)$$

Now for the left hand side of (10) we can take either (11) or (11.1). Equating (12) successively with (11) and (11.1), we have after transformation

$$\frac{1}{2r_1\left(2-\frac{r_1}{2}\cdot\frac{\alpha_2}{K_2}\right)}-\frac{\alpha_2}{2}\sqrt{1-\frac{4}{r_1}\cdot\frac{K_2}{\alpha_2}}\times\tan\left\{K_2\left(\sinh\ 2u_1-2u_1\right)-\frac{\pi}{4}+p_2\right\}$$

$$=-\left(\frac{\alpha_3}{2}-\frac{K_3}{r_1}\right)\times\tan\left(\frac{\alpha_3}{2}r_1-K_3\ \ln\frac{\alpha_3}{2}\ r_1-\frac{\pi}{4}+p_3\right)\ \ldots\ (13)$$

and

$$\left(\frac{\alpha_2}{2}-\frac{K_2}{r_1}\right)\times\tan\left(\frac{\alpha_2}{2}r_1-K_2\ \ln\ \frac{\alpha_2}{2}\ r_1-\frac{\pi}{4}+p_2'\right)$$

$$=\left(\frac{\alpha_3}{2}-\frac{K_3}{r_1}\right)\times\tan\left(\frac{\alpha_3}{2}r_1-K_3\ \ln\ \frac{\alpha_3}{2}\ r_1-\frac{\pi}{4}+p_3\right)\quad\ldots\ (13.1)$$

We cannot proceed further with the equations (13) and (13.1) without ascertaining the nature of the resultant field in the region 2. The scattering experiments of Rutherford (1927) with U_1 nucleus suggest that the resultant field in region 2 should in general be repulsive. Thus even if the total energy of the emitted alpha particle is wholly potential at r_0, its energy at r_1, should in general be partly potential and partly kinetic. If the particle comes out with the minimum energy, then evidently its kinetic energy ξ at r_0 is zero, and so we have at r_0,

$$(E_{\text{kin}})_{r_0}=\xi=(E_2)_{\min}+U(r_0)_{\xi=0}-\frac{2z^*e^2}{r_0}=0\qquad ..\quad (15)$$

and at r_1, since $U(r_1)=0$,

$$(E_{\text{kin}})_{r_1}=(E_2)_{\min}-\frac{2z^*e^2}{r_1}=\eta_0\qquad\ldots\quad (16)$$

where $(E_2)_{min}$ is the total energy in region 2, corresponding to $\xi=0$. But the alpha particle cannot always come out with the minimum energy, *i.e.* we cannot have always $\xi=0$ On the other hand, ξ may have as many discrete values as there are energy levels in the nucleus higher than the peak of the barrier. When the decaying nucleus emits the alpha particles from different energy levels, the product element will be left in the corresponding excited states. This product nucleus, like all other isomers, is likely to possess different spin values corresponding to its different excited states. Again as the interaction potential is of Yukawa type and spin dependent, $U(r)$ in (15) will change in magnitude with spin. Now, since the spin-difference is associated with the difference of energy states of the nucleus and hence with the initial k. e. ξ of the emitted alpha particle, $U(r)$ is evidently indirectly related to the initial kinetic energy of the alpha particle. Thus Eq (15) may be taken in the general form

$$E_2 + U(r_0)_\xi - \frac{2z^*e^2}{r_0} = \xi \qquad \dots (17)$$

On comparing (17) with (15) we have at once

$$E_2 = (E_2)_{min} + \xi + \beta \qquad \dots (18)$$

where

$$\beta = U(r_0)_{\xi=0} - U(r_0)_\xi \qquad \dots (18\ 1)$$

We shall, however, show later on that β is positive and consequently

$$U(r_0)_{\xi=0} = U(r_0)_{max} \qquad \dots (18.2)$$

Now since we have defined the second boundary r_1 as the distance at which the attraction potential is sensibly zero, it is reasonable to believe that for the small change β in $U(r_0)$, the change in the value of the potential at such a great distance r_1 is insignificant. Thus only the kinetic portion of the total energy of the alpha particle changes at the second boundary r_1, with any change in the initial kinetic energy ξ at r_0. Therefore, similar to (17), we should take Eq. (16) at r_1 in the general form

$$E_2 - \frac{2z^*e^2}{r_1} = \eta \qquad \dots (19)$$

where, [*vide* (18)]

$$\eta = \eta_0 + \xi + \beta \qquad \dots (19.1)$$

Now if S be the ratio of the straight mean $U(r_0)_\xi/2$ to the true mean U_0 for the attraction potential in region 2, we have, corresponding to Eq (17),

$$E_2 + U_0 - \frac{2z^*e^2}{\bar{r}} = \xi' \qquad \dots (20)$$

where

$$U_0 = \frac{U(r_0)_\xi}{2S} \qquad \dots (20.1)$$

Comparing Eq. (20) and (17), U_0 is seen to be the actual attraction potential at $\bar{r}$, where ξ' is the corresponding kinetic energy. Again, since $U(r)$ is of Yukawa type, the mean value U_0, given by (20.1) and which has been taken constant for the wave equation (1.4) or (2), must be less than the straight average $U(r_0)_\xi/2$. Hence in Eq. (20.1) we have

$$s > 1 \qquad \ldots \quad (21)$$

It will be shown later on [*vide* Eq. (32.2)] that $s < 2$. It is obvious that for the lower limit $s=1$, ξ' and $\bar{r}$ in (20) are the averages of the corresponding quantities at the two boundaries r_0 and r_1. It may be easily seen that for the upper limit $s=2$, we have

$$\bar{r} = \frac{4r_0 r_1}{3r_0 + r_1} \qquad \ldots \quad (20.2)$$

$$\xi' = (3\eta + \xi)/4 \qquad \ldots \quad (20.3)$$

Now having thus defined U_0 for the second region we get at once from (2.2) and (2.4)

$$\frac{\alpha_2}{K_2} = \frac{2(E_2 + U_0)}{z^* e^2} \qquad \ldots \quad (22)$$

Now, substituting in the equation of continuity (13) the values of α_2/K_2, α_2, and r_1 from (22), (2.2) and (19) respectively, we have after transformation

$$\frac{E_2-\eta}{2(U_0+\eta)} \cdot \frac{1}{\tan\left\{K_2(\sinh 2u_1 - 2u_1) - \frac{\pi}{4} + p_2\right\}} + \frac{8\pi z^* c^2 \sqrt{2M(U_0+\eta)}}{h(E_2-\eta)}$$

$$= \frac{8\pi z^* c^2 \sqrt{M}}{h\sqrt{2E_2}} \frac{(E_2+\eta)}{(E_2-\eta)} \cdot \left\{ \frac{\tan\left(\frac{\alpha_3}{2} r_1 - K_3 \ln \frac{\alpha_3}{2} r_1 - \frac{\pi}{4} + p_3\right)}{\tan\left\{K_2(\sinh 2u_1 - 2u_1) - \frac{\pi}{4} + p_2\right\}} \right\} \quad \ldots \quad (23)$$

Similarly from (2.2), (2.4) and (19), the alternative equation (13.1) transforms into

$$\frac{2\sqrt{1+U_0/E_2}}{(1-\eta/E_2)} - \frac{1}{\sqrt{1+U_0/E_2}} = \frac{(1+\eta/E_2)}{(1-\eta/E_2)} \cdot \frac{\tan\left(\frac{\alpha_3}{2} r_1 - K_3 \ln. \frac{\alpha_3}{2} r_1 - \frac{\pi}{4} + p_3\right)}{\tan\left\{\frac{\alpha_2}{2} r_1 - K_2 . \ln \frac{\alpha_2}{2} r_1 - \frac{\pi}{4} + p_2'\right\}} \quad \ldots \quad (23.1)$$

Now from the optical analogy, that the transmitted waves do not suffer any phase change at a boundary, it is reasonable to assume that the matter waves do not undergo any change of phase at the boundary r_1. Hence the ratio of the tangents of the phase angles at r_1, on the right hand side of both equations (23) and (23.1) become 1, and so we have

$$\left.\begin{aligned} &\frac{(E_2-\eta)}{2(U_0+\eta)} \cdot \frac{1}{\tan\left\{K_2(\sinh 2u_1 - 2u_1) - \frac{\pi}{4} + p_2\right\}} + \frac{8\pi z^* e^2 \sqrt{2M(U_0+\eta)}}{h(E_2-\eta)} \\ &\qquad = \frac{8\pi z^* c^2 \sqrt{M}(E_2+\eta)}{h\sqrt{2E_2}\,(E_2-\eta)} \end{aligned}\right\} \quad (23.2)$$

and
$$\frac{2\sqrt{1+U_0/E_2}}{(1-\eta/E_2)} - \frac{1}{\sqrt{1+U_0/E}} = \frac{(1+\eta/E_2)}{(1-\eta/E_2)} \quad \ldots \quad (23.3)$$

On solving (23.3) for $\sqrt{1+\frac{U_0}{E_2}}$, it may be easily shown that either

$$U_0=0, \text{ or} = \left\{\frac{1}{4}\left(1-\frac{\eta}{E_2}\right)-1\right\} \times E_2 \quad \ldots \quad (23.4)$$

The second value of U_0 in (23.4) is negative, as $\frac{\eta}{E_2} < 1$. Thus both the possible values of U_0 obtained in (23.4) should be rejected, as going against our fundmental conception of U_0. We are thus left with the only equation of continuity (23.2)

To find the unknown epoch p_2 in (23.2), we consider the behaviour of the solution R_2, given in Eq. (8.1) for the region 2(b), i.e. for $r' < r < r_1$. Since r' is not a boundary, R_2 should be valid at r'. At $r=r'$, the real part of the solution R_2 in (8.1) as given in (10.1) is

$$R_2=\left| \frac{\text{const.}}{r} . \sqrt{\coth u} \ . \ \cos\left\{ K_2(\sinh 2u-2u) - \frac{\pi}{4} + p_2\right\}\right|_{r=r'} \quad \ldots \quad (23.5)$$

where
$$(\coth u)_{r=r'} = \sqrt{\frac{1}{1-4K_2/\alpha_2 r'}}$$

Again from (2.2), (2.4) and (7.2) we have $\frac{4K_2}{\alpha_2 r'}=1$ Therefore

$$(\coth u)_{r=r'} = \infty \quad \ldots \quad (24)$$

i.e., the solution R_2 in (23.5) is infinite, which is impossible. Therefore in order to make the solution bounded at r', we must take in (23.5)

$$\cos\left\{K_2(\sinh 2u-2u) - \frac{\pi}{4} + p_2\right\}\Big|_{r=r'} = 0$$

$$K_2(\sinh 2u-2u) - \frac{\pi}{4} + p_2\Big|_{r=r'} = \frac{\pi}{2} \quad \ldots \quad (24.1)$$

From Eq. (24), $|u|_{r=r'}=0$ and $(\sinh 2u)_{r=r'}=0$. Hence from (24.1) we at once get

$$p_2=3 . \frac{\pi}{4} \quad (25)$$

Using (25) in (23.2) we have finally for the continuity equation

$$\frac{E_2-\eta}{2(U_0+\eta)} . \frac{1}{\tan\left\{ K_2(\sinh 2u_1-2u_1)+\frac{\pi}{2}\right\}} + \frac{8\pi z^* e^2\sqrt{2M(U_0+\eta)}}{h(E_2-\eta)} = \frac{8\pi z^* e^2(E_2+\eta)}{hv(E_2-\eta)} \quad (26)$$

where v is the observed velocity of alpha particle, given by $E_2 = 1/2.Mv^2 = E_3$. The first term in (26) is very small compared to the second and the third, as can be seen by putting the approximate values for all the quantities involved. So neglecting this term, to a first approximation we find

$$\frac{U_0}{E_2} = \frac{1}{4} - \frac{\eta}{2E_2} + \frac{\eta^2}{4E_2^2} \quad \cdots \quad (26.1)$$

Now using (26.1), the correction term in (26) becomes

$$2\left(1 - \frac{3\eta}{E_2}\right) \Big/ \tan\theta \quad \cdots \quad (27)$$

where
$$\theta = \pi \left\{ \frac{z^* \times 0.1393(\sinh 2u_1 - 2u_1)}{v - 10^{-9} \times \sqrt{\frac{5}{4} - \frac{\eta}{2E_2}}} + 0.5 \right\} \quad \cdots \quad (27.1)$$

neglecting the small $\frac{\eta^2}{E_2^2}$ term.

Again to find $2u_1$ and $\sinh 2u_1$ in (27.1) we note that by definition

$$\cosh^2 u_1 = \frac{\alpha_2 r_1}{4K_2} = \frac{1 + U_0/E_2}{1 - \eta/E_2} = \frac{5/4 - \frac{\eta}{2E_2}}{1 - \eta/E_2} \quad \cdots \quad (27.2)$$

from which $2u_1$ and $\sinh 2u_1$ can be calculated if η/E_2 is known. Substituting (27) in (26) we have

$$\frac{8\pi z^* e^2 \sqrt{2M(U_0 + \eta)}}{h(E_2 - \eta)} = \frac{8\pi z^* e^2 (E_2 + \eta)}{hv(E_2 - \eta)} - \frac{2\left(1 - \frac{3\eta}{E_2}\right)}{\tan\theta} \quad \cdots \quad (28)$$

Therefore

$$U_0 + \eta = \frac{1}{8} Mv^2\left(1 + \frac{2\eta}{E_2}\right) - \frac{Mv^3\hbar\left(1 - \frac{3\eta}{E_2}\right)}{8z^* e^2 \tan\theta} + \text{higher term}$$

or
$$U_0 + \eta/2 = \left\{ \frac{1}{4} - \frac{\hbar v\left(1 - \frac{3\eta}{E_2}\right)}{4z^* e^2 \tan\theta} \right\} \times E_2 \quad \cdots \quad (29)$$

Determination of β

At the outset it need be pointed out that since β is involved only in U_0, Eq. (29) is unaffected even if we take $\beta = 0$ always In that case, however, we have to assume with Gamow and others that the size of a nucleus of same z^* fluctuates as it is raised to different energy states by emitting different α-groups. Thus we have to assume as many as 13 radii for the same RaF nucleus. This seems to us less plausible than our conception of constant r_0, which follows from (29) with discrete β-values. So let us now decide whether β is positive or negative.

Let us first assume that β is negative, and we have then from (20.1) and (18.1)

$$U_0 = \frac{[U(r_0)]_{\xi=0}}{2s} + \frac{\beta}{2s} \quad \dots \quad (30)$$

It is evident from (30) that when $\beta = 0$, U_0 is minimum. Hence

$$(U_0)_{\min} = [U(r_0)]_{\xi=0}/2s.$$

$$(U_0)_{\max} = [U(r_0)]_{\xi=0}/2s + \frac{\beta_{\max}}{2s}.$$

and so

$$(U_0)_{\max} - (U_0)_{\min} = \beta_{\max}/2s \quad \dots \quad (30.1)$$

Again when $\beta = 0$, $\xi = 0$, therefore $(U_0)_{\min}$ corresponds to $(E_2)_{\min}$ and so in that case $\eta = \eta_0$ [vide (19.1).] Also when β is maximum, ξ is also maximum, and $(U_0)_{\max}$ corresponds to $(E_2)_{\max}$, and in that case $\eta = \eta_{\max}$. Therefore neglecting the small correction term $3\eta/E_2$ in (29) we have similar to (30.1).

$$(U_0)_{\max} - (U_0)_{\min} = \frac{1}{4}\left((E_2)_{\max} - (E_2)_{\min} - \frac{\eta_{\max} - \eta_0}{2}\right) - \frac{\hbar}{4z^*e^2}\left\{\frac{v_{\max}(E_2)_{\max}}{\tan\theta_0} - \frac{v_{\min}(E_2)_{\min}}{\tan\theta_{\max}}\right\} \quad \dots \quad (31)$$

Again from (19.1) and (18) for negative β

$$\eta_{\max} - \eta_0 = (E_2)_{\max} - (E_2)_{\min} = \xi_{\max} - \beta_{\max} \quad \dots \quad (31.1)$$

Now substituting (31.1) in (31)

$$(U_0)_{\max} - (U_0)_{\min} = -\frac{1}{4}(\xi_{\max} - \beta_{\max}) - \frac{\hbar}{4z^*e^2}\left\{\frac{v_{\max}(E_2)_{\max}}{\tan\theta_0} - \frac{v_{\min}(E_2)_{\min}}{\tan\theta_{\max}}\right\} \quad \dots \quad (31.2)$$

From (30.1) and (31.2) we have

$$\beta_{\max} = \frac{s}{(s-2)}\xi_{\max} + \frac{\hbar s}{z^*e^2(s-2)}\left\{\frac{v_{\max}(E_2)_{\max}}{\tan\theta_0} - \frac{v_{\min}(E_2)_{\min}}{\tan\theta_{\max}}\right\} \quad \dots \quad (32)$$

Since in (30) β is assumed to be negative, $\beta_{\max}$ in (32) should be positive. So the right hand side of (32) should also be positive, *i.e.*, $s > 2$. Again, if β is negative, $\xi_{\max} - \beta_{\max} = (E_2)_{\max} - (E_2)_{\min}$, (*vide* 18). So $\xi_{\max} > \beta_{\max}$. Therefore we have in (32), $\frac{s}{s-2} < 1$ which is impossible if $s > 2$. Thus β cannot be negative. Hence β is positive which means, as assumed in (18.2), that the attraction potential $U(r)$ becomes higher as the nucleus is raised to higher excited states due to emission of alpha particles at lower energies. Thus it it follows that $U(r)$ or U_0 is related indirectly to the kinetic energy of the outgoing alpha particle as already mentioned.

Now since β is positive, we shall have instead of (32)

$$\beta_{\max} = \frac{s}{(2-s)}\xi_{\max} + \frac{\hbar s}{z^*e^2(2-s)}\left\{\frac{v_{\max}(E_2)_{\max}}{\tan\theta_0} - \frac{v_{\min}(E_2)_{\min}}{\tan\theta_{\max}}\right\} \quad \dots \quad (32.1)$$

From (32.1) and (21) we have evidently

$$1 < s < 2. \qquad \dots \quad (32.2)$$

Since from (18), $\xi_{max} + \beta_{max} = (E_2)_{max} - (E_2)_{min}$ = range of α-energy spectrum we have from (32.1)

$$\beta_{max} = \frac{s}{2}(E_2)_{max} - (E_2)_{min} + \frac{\hbar s}{2z^* e^2}\left\{\frac{v_{max}\,(E_2)_{max}}{\tan\theta_0} - \frac{v_{min}\,(E_2)_{min}}{\tan\theta_{max}}\right\} \qquad \dots \quad (32.3)$$

and
$$\xi_{max} = \frac{(2-s)}{2}(E_2)_{max} - (E_2)_{min} - \frac{\hbar s}{2z^* e^2}\left\{\frac{v_{max}\,(E_2)_{max}}{\tan\theta_0} - \frac{v_{min}\,(E_2)_{min}}{\tan\theta_{max}}\right\} \qquad \dots \quad (32.4)$$

Now it follows from (32.3) and (32.4) that in the case of the elements which do not show alpha energy spectrum, $\xi = \beta = 0$. These elements possess only one effective energy level higher than the peak of the potential barrier. There are, however, other energy states lower than the peak of the barrier. These levels being of insufficient energy are incapable of emitting alpha particles. With smaller values of z, the potential barrier becomes less high and consequently the number of alpha transformations giving rise to spectrum will increase, and necessarily the range of the alpha energy spectrum becomes wider. It is remarkable that the present theory thus provides a qualitative explanation for the observed fact that as we go up a radioactive series, z increases, the number of spectral lines in general decreases till it becomes one (*vide* Table I). Irregularities, however, must be due to some structural complexities

EVALUATION OF NUCLEAR RADIUS r_0

Since β is positive, writing for $U(r_0)$ from (18.1) and (18.2), we have from (20.1)

$$U_0 = \frac{U(r_0)_{max}}{2s} - \frac{\beta}{2s} \qquad \dots \quad (33)$$

Therefore
$$(U_0)_{min} = \frac{U(r_0)_{max}}{2s} - \frac{\beta_{max}}{2s} \qquad \dots \quad (33.1)$$

Since β_{max} corresponds to ξ_{max} and hence $(E_2)_{max}$, we have from (29)

$$(U_0)_{min} = \left\{\frac{1}{4} - \frac{\hbar v_{max}\,(1 - 3\eta_{max}/(E_2)_{max})}{4z^* e^2 \tan\theta_0}\right\} \times (E_2)_{max} - \frac{\eta_{max}}{2} \qquad \dots \quad (34)$$

Again from (15) and (18.2), we have

$$r_0 = \frac{2z^* e^2}{(E_2)_{min} + [U(r_0)]_{max}} \qquad \dots \quad (35)$$

On substituting for $U(r_0)_{max}$ from (33.1) and using (34) we get

$$U(r_0)_{max} = \left\{\frac{s}{2} - \frac{s\hbar v_{max}\,(1 - 3\eta_{max}/(E_2)_{max})}{z^* e^2 \tan\theta_0}\right\} \times (E_2)_{max} - s\left(\eta_{max} - \frac{\beta_{max}}{s}\right) \dots \quad (35.1)$$

TABLE I

Name of decaying element	z^*	No of α-energy groups	obsd. $(E_2)_{max}$ Mev.	obsd. $(E_2)_{min}$ Mev.	$U(r_0)_{max}$ Mev.	Nuclear radius $r_0 \times 10^{+12}$ cm.	Produced by
U_1^{238}	90	1	4.210		4 360	3 024	
U_{11}^{234}	90	1	4.780		3 473	3 139	$_{91}UX_2^{234}$ β-decay
Io^{230}	88	1	4.810		3 409	3 083	$_{92}U_{11}^{234}$ α-decay
Ra^{226}	86	7	4 793	3 947	3 629	3.765	$_{90}Io^{230}$ α-decay
Rn^{222}	84	1	5.486		3 674	2 640	$_{88}Ra^{226}$ α-decay
Ra A^{218}	82	1	5.998		3 890	2 380	$_{86}Rn^{222}$ α decay
Ra C'^{214}	82	13	10.509	7.683	3 526	2 106	$_{83}RaC^{214}$ β-decay
Ra F^{210}	82	13	5 303	3.685	4 327	2.947	$_{83}RaE^{210}$ β-decay
Ra C^{211}	81	3	5.517	5 333	3 781	2 560	$_{82}RaB^{214}$ β-decay
Th^{232}	88	1	4.20		3 384	3 341	...
Rd Th^{228}	88	2	5 418	5 335	3.784	2.779	$_{89}MsTh_{11}^{228}$ β-decay
Th X^{224}	86	1	5 680		3 814	2 608	$_{90}RdTh^{228}$ α-decay
Tn^{220}	84	1	6 280		4.077	2.334	$_{88}ThX^{224}$ α-decay
Th A^{216}	82	1	6 774		4 222	2.146	$_{86}Th^{220}$ α-decay
Th C'^{212}	82	3	10 553	8.776	7.50	1 450	$_{83}ThC^{212}$ β-decay
Th C^{212}	81	6	6 054	5 601	4 151	2 392	$_{82}ThB^{212}$ β decay
Ac U^{235}	90	1	4.330		3.643	3 249	
Pa^{231}	89	1	5 00		3.523	3 007	$_{90}UY^{231}$ β-decay
Rd Ac^{227}	88	11	6.051	5 674	5.513	2 265	$_{89}Ac^{227}$ β-decay
Ac X^{223}	86	3	5 720	5 533	3 896	2 626	$_{90}$Rd Ac^{227} α-decay
An^{219}	84	3	6.826	6 436	4.449	2.221	$_{88}$Ac X^{223} α-decay
Ac A^{215}	82	1	7 365		4.381	2.009	$_{86}An^{219}$ α decay
Ac C'^{211}	82	1	7 434		4.390	1.997	$_{83}$Ac C^{211} β-decay
Ac C^{211}	81	2	6.619	6 262	4 273	2.213	$_{82}$Ac B^{211} β-decay

Now from (35) and (35.1) we get after transformation

$$r_0 = \frac{4z^{*2}c^4 \tan\theta_0}{(E_2)_{max}\{(s+2)z^*e^2 \tan\theta_0 - s\hbar v_{max}(1 - 3\eta_{max}/(E_2)_{max})\} - L} \tag{36}$$

where $\tan\theta_0$ corresponds to v_{max} and is given in (27.1), and

$$L = 2z^*e^2 \tan\theta_0 (s\eta_{max} + \xi_{max})$$

from (32.4) neglecting small term involving $\hbar$, we have

$$L = 2z^* e^2 \tan\theta_0 \left(s\eta_{max} + \frac{(2-s)}{2} \text{ range} \right) \quad \ldots \quad (36.1)$$

For the elements which do not show energy spectrum (36 1) takes the simple form

$$L = 2sz^* e^2 \eta_0 \tan\theta_0 \quad \ldots \quad (36.2)$$

It should be noted that the formula for r_0 in (36) contains s and η_0, yet to be found The value of η_0 will be obtained from the formula to be deduced later on (*vide* 40). Hence we are in a position to calculate r_0 for different nuclei, if we take at present for s its limiting value† $s=2$ (vide 32 2) In addition to the size of the nucleus, information regarding the limiting values of the attraction potential $U(r)$ at r_0, are also known (*vide* 35 1). It is remarkable that the magnitude of the short range potential calculated from our theory ranges over several Mev which agrees with the order of the magnitude obtained from other methods (e.g. nuclear binding energy calculation from mass defects, etc) Table I gives the calculated values of r_0, $U(r_0)_{max}$, for the elements of all the three radioactive series with $s=2$.

It is to be noted that Eq. (36) gives an *explicit* formula for the size of a nucleus. Previously, the values of nuclear radii were obtained from an *implicit* $\log\lambda - E$ relation. This method seems to be only approximate because, (1) the $\log \lambda - E$ relation, given by the leakage theory, is open to the objections raised by Chang (1948), Berthellot (1942) and others, (2) this $\log \lambda$-E relation really contains an unknown averaging (normalizing) factor, and in calculating r_0, all the previous workers, instead of finding that unknown factor, have taken it to be unity, and (3) these r_0 values are calculated from a relation in which λ and E are variable, so that in the case of energy spectrum r_0 should be different for the different values of the observed energy for the same z^*. Thus, for example, for the element RaC′, there should be at least 13 values of nuclear radii for the same element, which appears to be improbable.

On the other hand, our explicit formula for r_0 is *independent* of λ, and comes out in terms of constant quantities, such as z^*, M, e, $(E_2)_{max}$, etc.

Determination of η

From what has been discussed under Eq. (16) and also from the Eq. (18.1), it is seen that as ξ at r_0 increases, $U(r_0)$ decreases. Similarly at distance $\bar{r}$, the attraction potential U_0 may also be taken related in the same way with the corresponding kineting energy of the alpha particle $\xi' = (\xi + 3\eta)/4$. [*vide* (20 3)]. This suggests a simple relationship between the two and we take tentatively

$$U_0 \times (\xi + 3\eta)^n = F(\xi) \quad \ldots \quad (37)$$

where n is an unknown constant and F is a slowly varying function of ξ.

† Numerical calculations suggest that s should be very close to 2.

A constant F leads to absurd result, as can easily be seen by comparing the two extreme cases of (37), namely $\xi = \xi_{max}$ Now for $\xi = \xi_{max}$ we have from (37)

$$(U_0)_{max} \times (\xi + 3\eta)^n{}_{max} = F(\xi_{max}) \qquad \ldots \quad (37.1)$$

Or, using (33.1) and (19.1)

$$\frac{U(r_0)_{max}}{2s} - \frac{\beta_{max}}{2s} = \frac{F(\xi_{max})}{(4\xi_{max} + 3\eta_0)^n \left\{ 1 + \dfrac{3\beta_{max}}{4\xi_{max} + 3\eta_0} \right\}^n} \qquad \ldots \quad (37.2)$$

Since $3\beta_{max} < (4\xi_{max} + 3\eta_0)$, we have from (37.2)

$$\frac{U(r_0)_{max}}{2s} - \frac{\beta_{max}}{2s} = \frac{F(\xi_{max})}{(4\xi_{max} + 3\eta_0)^n} - \frac{6\ n\ sF(\xi_{max})/(4\xi_{max} + 3\eta_0)^n}{(4\xi_{max} + 3\eta_0)} \cdot \frac{\beta_{max}}{2s} \qquad \ldots \quad (38)$$

Again, since $\beta_{max} \ll U(r_0)_{max}$, we have from (38), on equating terms of the same order

$$\frac{U(r_0)_{max}}{2s} = \frac{F(\xi_{max})}{(4\xi_{max} + 3\eta_0)^n} \qquad \ldots \quad (38.1)$$

and

$$\frac{6ns\ F(\xi_{max})/(4\xi_{max} + 3\eta_0)^n}{(4\xi_{max} + 3\eta_0)} = 1 \qquad \ldots \quad (38.2)$$

Substituting (38.1) in (38.2) we have

$$3nU(r_0)_{max} = 4\xi_{max} + 3\eta_0 \qquad \ldots \quad (39)$$

Now putting $U(r_0)_{max}$ from (33.1) and using (34), (32.3) and (32.4), we have after transformation, if $sn = s'$

$$n_0 = \frac{s'}{2(s'+1)} \cdot (E_2)_{max} - \frac{(3s' - 4s + 8)}{6(s'+1)} \cdot [(E_2)_{max} - (E_2)_{min}]$$

$$\cdot \frac{2\hbar s\left(\dfrac{4+3n}{4} \dfrac{v_{min}(E_2)_{min}}{\tan\theta_{max}} - \dfrac{v_{max}(E_2)_{max}}{\tan\theta_0} \right)}{3(s'+1)z^* e^2} \qquad \ldots \quad (39.1)$$

It may be shown from Eq. (27) that if we assume $\eta \geq \frac{1}{3}E$, we have impossible results. Let us first put $\eta = \frac{1}{3}E$. In that case Eq (27) or the first term on the left hand side of (23) becomes zero, *i.e.*, we shall have $2K_2 = 0$, which is impossible [*vide* (2.4).] Again if $\eta > \frac{1}{3}E$, the numerator of (27) becomes negative and so the term on the left hand side of (13) becomes positive, which is impossible. Hence we conclude that $\eta < \frac{1}{3}E$. Using this condition we can easily find from (39.1) that for the elements which do not show alpha energy spectrum s' is certainly less than 2. Again since lower limit of $\eta = 0$, for the average value of η, $s' = 1/2$. With this value of s', in the general case, we have from (39.1) neglecting the higher term

$$\eta_0 = \frac{1}{6} (E_2)_{max} - \frac{1}{6}\left((E_2)_{max} - (E_2)_{min} \right) \qquad \ldots \quad (40)$$

and hence

$$\eta_{max} = \frac{1}{6}(E_2)_{max} + \frac{5}{6}\left((E_2)_{max} - (E_2)_{min} \right) \qquad \ldots \quad (40.1)$$

SECTION 2

Disintegration constant λ

We now consider the condition of continuity at the first boundary $r=r_0$. This condition of continuity gives a formula for the viscosity coefficient b, from which we shall deduce the disintegration constant λ.

Since near r_0, $K'_2/x > 1$, we have to take the solution (8) for the region just outside r_0. In order to deduce the proper sign in the exponential, we consider the behaviour of the solution at r', upto which it is valid. Now at $r=r'$, cot u in Eq. (8) is from (6.1)

$$(\cot u)_{r=r'} = \frac{1}{\sqrt{(4K_2/a_2 r')-1}} = \infty$$

since from (2.2), (2.4) and (7.2) $4K_2/a_2 r' = 1$.

Therefore in order to make the solution bounded at r', we shall have to take

$$e^{\pm K_2(2u'-\sin 2u')}=0 \qquad \ldots \quad (41)$$

Now since $\cot u=\infty$, $u=n\pi$, where $n=0, 1, 2, 3$ etc. But if we put $n=0$, we get $e^{\pm K_2(2u'-\sin 2u')}=1$, and the function becomes infinite both for positive and negative sign of the exponential. Hence $n=1$ at least. As K_2 is large, for $n=1, 2, 3$ etc., the condition (41) is satisfied only if the negative sign is taken. Thus Eq. (8) becomes

$$R_2=\frac{\text{const.}}{r}\cdot\sqrt{\cot u}\cdot e^{-K_2(2u-\sin 2u)} \qquad \ldots \quad (42)$$

Now the boundary conditions at $r=r_0$ give

$$\left|\frac{\delta}{\delta r}\cdot\log.\frac{\sin\alpha_1 r}{r}\right|_{r=r_0}=\left|\frac{\delta}{\delta r}\log\frac{\sqrt{\cot u}.e^{-K_2(2u-\sin 2u)}}{r}\right|_{r=r_0} \ldots \quad (43)$$

The left hand side of (43) is $\alpha_1\cot\alpha_1 r_0-\frac{1}{r_0}$,

where
$$\alpha_1^2=\frac{8\pi^2 ME_1}{h^2}\left(1+\frac{b^2h^2}{16\pi^2E_1^2}\right).$$

Since
$$\frac{b^2h^2}{16\pi^2E_1^2} << 1,$$

we have

$$\alpha_1 r_0=\sqrt{\frac{8\pi^2ME_1}{h^2}}\,r_0+\sqrt{\frac{8\pi^2ME_1}{h^2}}\cdot\frac{b^2h^2}{3^2\pi^2E_1^2}\cdot r_0=A+B\ \text{(say)} \quad \ldots \quad (43.1)$$

So

$$\alpha_1\cot\alpha_1 r_0=\alpha_1\cot(A+B)=\alpha_1\cot A-\alpha_1 B-\alpha_1 B\cot^2 A+B^2\alpha_1\cot A.$$

Now since A is of the order of 40, and cot A of the order 4, by putting $\cot A=\delta/A$ and the values of A and B from (43.1) and on neglecting terms containing b^4 and b^6 being too small, we have after transformation,

$$\alpha_1 \cot \alpha_1 r_0 - \frac{1}{r_0} = -\frac{1}{r_0} + \frac{1}{r_0}\left\{\delta - b^2\left(\frac{\delta(\delta-1)\hbar^2}{32\pi^2 E_1^2} + \frac{Mr_0^2}{4E_1}\right)\right\} \quad \ldots \quad (43.2)$$

Now the right hand side of (43) is

$$-\frac{1}{r_0} + \frac{1}{4r_0(1-\alpha_2 r_0/4K_2)} + \frac{\alpha_2}{2}\sqrt{\frac{4K_2}{\alpha_2 r_0} - 1} \quad \ldots \quad (43.3)$$

Equating (43.2) with (43.3) we have

$$\delta - b^2\left(\frac{Mr_0^2}{4E_1} + \frac{\delta'(\delta-1)\hbar^2}{8E_1^2}\right) = \frac{1}{2\left(2 - \frac{r_0}{2}\cdot\frac{\alpha_2}{K_2}\right)} + \frac{\alpha_2 r_0}{2}\cdot\sqrt{\frac{4}{r_0}\cdot\frac{K_2}{\alpha_2} - 1} \quad \ldots \quad (44)$$

Now from (22) and (20), and using (20.2) for $\bar{r}$, we get

$$\frac{\alpha_2}{K_2} = 4(1 + \xi'\bar{r}/2z^*e^2)/\bar{r} \quad \ldots \quad (44.1)$$

and

$$\frac{\alpha_2 r_0}{2} = \frac{K_2}{2r_1}(3r_0 + r_1)\left(1 + \frac{\xi'\bar{r}}{2z^*e^2}\right) \quad \ldots \quad (44.2)$$

Therefore the right hand side of (44) becomes after transformation

$$\frac{r_1}{3\left\{r_1 - r_0 - \frac{2\xi' r_0 r_1}{3z^*e^2}\right\}} + K_2\left\{1 - r_0^2\left(\frac{3}{2r_1} - \frac{1}{2r_0} + \frac{\xi'}{z^*e^2}\right)^2\right\}^{\frac{1}{2}} \quad \ldots \quad (45)$$

On combining (45) and (44), on neglecting 1 compared to 3δ, we get

$$b^2\left(\frac{Mr_0^2}{4E_1} + \frac{\delta(\delta-1)\hbar^2}{8E_1^2}\right) = \delta - K_2\left\{1 - \frac{r_0^2}{r_1^2}\left(\frac{3}{2} + \frac{\xi' r_1}{z^*e^2} - \frac{r_1}{2r_0}\right)^2\right\}^{\frac{1}{2}}$$

Since again $\xi' \ll z^* e^2/r_1$, we have finally for b, on putting $r_0^2/r_1^2 = n'$, and simplifying

$$b = \frac{v'}{r_0}\cdot\frac{\sqrt{2\delta - K_2\left\{2 - \frac{n'}{2}\left(3 - \frac{1}{\sqrt{n'}}\right)^2\right\}^{\frac{1}{2}}}}{\sqrt{1 + \delta^2\hbar^2/M^2 v'^2 r_0^2}} \quad \ldots \quad (46)$$

Now we have by definition

$$b = \frac{1}{D_s}\cdot\frac{dD_s}{dt} = \frac{1}{N_s}\cdot\frac{dN_s}{dt} \quad \ldots \quad (47)$$

where D_s is the probable number density at the surface and N_s, is the probable number of α-particles in a thin shell of width Δr_0, just inside the surface. Again

$$\frac{dN_s}{dt} = \frac{dN_a}{dt} = \frac{1}{N}\frac{dN}{dt} = \lambda \quad \ldots \quad (48)$$

where N_α is the total number of alpha particles inside a given nucleus, and N, the total number of nuclei in the sample. Therefore from (47), (48), and (46) we have

$$\lambda = b.N_s = \frac{v'}{r_0} \cdot \frac{\sqrt{2\delta - K_2\left\{2 - \frac{n'}{2}\left(3 - \frac{1}{\sqrt{n'}}\right)^2\right\}^{\frac{1}{2}}}}{\sqrt{1 + \delta^2\hbar^2/M^2v'^2r_0}} \cdot \frac{C_1^2 \sin^2 \alpha_1 r_0}{2} \quad 4\pi r_0^2 \Delta r_0$$

or, from (42)

$$\lambda = C \cdot \frac{\sqrt{2\delta - K_2\left(2 - \frac{n'}{2} \cdot \left(3 - \frac{1}{\sqrt{n'}}\right)^2\right)^{\frac{1}{2}}}}{\sqrt{1 + \delta^2\hbar^2/M^2v'^2r_0^2}} \cdot \frac{v'}{r_0} \quad \cot u_0 . e^{-2K_2(2u_0 - \sin 2u_0)} \tag{49}$$

where

$$C = 4\pi C_2^2 \Delta r_0.$$

and C_1 and C_2 are the averaging factors for the solution inside and just outside r_0.

In the expression for λ, the exponential part is important However, it requires to be mentioned that in the exponential of our formula, unlike the previous formulae, we have in K_2, not the observed energy E_2, but $E_2 + U_0$. A study of Chang's curve will show that if log $\lambda - E$ curve is plotted against the internal energy, *i.e.*, $E_2 + U(r_0)$, the discrepancy can be resolved

The quantitative agreement of our formula for λ will be considered later on, only when C in Eq. (49.1) and s' in (39.1) can be known exactly.

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